Probability module Day at-home exercises

(These exercises are taken from de Groot.)

1. In a certain city, 30% of the people are Republicans, 50% are Democrats, and 20% are Independents. Records show that in a particular election, 65% of the Republicans voted, 82% of the Democrats voted, and 50% of the Independents voted. If a person in the city is selected at random and it is learned that she did not vote in the last election, what is the probability that she is a Democrat?

2. Suppose that a box contains five coins, and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed (i=1…5), and suppose that p_1=0, p_2=1/4, p_3=1/2, p_4=3/4, and p_5=1.
    a. Suppose that one coin is selected at random from the box and that when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected, for i=1…5.

    b. If the same coin were tossed again, what would be the probability of obtaining another head?

c. If a tail had been obtained on the first toss of the selected coin and the same coin were tossed again, what would be the probability of obtaining a head on the second toss?

3. A bag contains C_1 coins of type 1, C_2 coins of type 2, and C_3 coins type 3. A coin is pulled out of the bag at random and flipped 100 times, producing *h* heads and 100-*h* tails. Let T be a random variable representing the type of the coin that was pulled out ($T \in \{1, 2, 3\}$). Each type of coin has a different probability of flipping heads: p1=Pr(heads|T=1), p2=Pr(heads|T=2), and p3=Pr(heads|T=3). Using Bayes' rule, write a formula for the posterior probabilities of the coin types after you've seen the outcome from the 100 flips: Pr(T=1| *h* heads in 100 flips), Pr(T=2| *h* heads in 100 flips), and Pr(T=3| *h* heads in 100 flips). After using Bayes' rule, you'll need to use the binomial distribution.

4. Suppose a coin has an unknown probability θ of producing heads and that it has produced x heads in N flips. The maximum likelihood estimate of θ, θ_{MLE}, is x/N. Derive this from the likelihood function:

$$L(\theta) = \Pr(x \mid N, \theta) = \binom{N}{x} \theta^x \left(1-\theta\right)^{N-x}$$

This is best done in 3 steps.

1. Instead of finding θ that maximizes the likelihood directly, find the θ that maximizes the logarithm of the likelihood. Since the logarithm is a monotonically increasing function, the highest point of the likelihood is also the highest point of the log of the likelihood.
2. Compute the derivative of the log likelihood with respect to θ, equate it to zero, and solve the equation for theta. Where the derivative is zero, the likelihood is at extremum -- either a minimum or a maximum.
3. There are two ways to show that this extremum is a maximum.
    a. Compute the second derivative of the log likelihood and show that it is negative at the extrememum, so the curve is concave down there.
    b. Probably easier, show that neither of the two limits of the interval, θ=0 or θ=1, is a maximum. Since there is only one extremum between those limits, it must be the maximum.